Applications of the Quadratic Assignment Problem

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OPTIMIZATION AND SYSTEMS ENGINEERING

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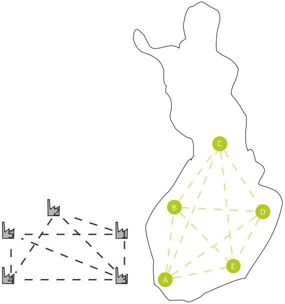
- Introduced by Koopmans and Beckmann in 1957
- Cited by ≈ 1500
- Among the hardest combinatorial problems
- Real and test instances easily accessible (QAPLib - A quadratic assignment problem library)
- Instances of size $N = 30$ are still unsolved



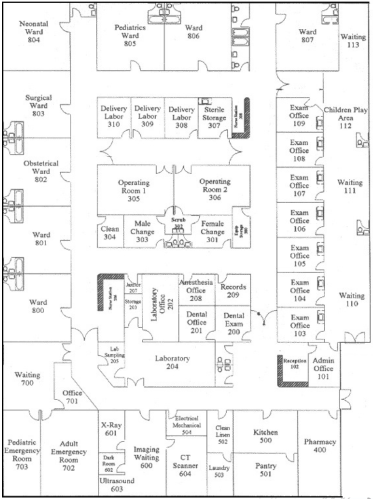
- Optimal assignment of factories to the cities marked in green
- Distances between the cities and flows between the factories shown below

$$A = \begin{bmatrix} 0 & 3 & 6 & 4 & 2 \\ 3 & 0 & 2 & 3 & 3 \\ 6 & 2 & 0 & 3 & 4 \\ 4 & 3 & 3 & 0 & 1 \\ 2 & 3 & 4 & 1 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 10 & 15 & 0 & 7 \\ 10 & 0 & 5 & 6 & 0 \\ 15 & 5 & 0 & 4 & 2 \\ 0 & 6 & 4 & 0 & 5 \\ 7 & 0 & 2 & 5 & 0 \end{bmatrix}$$

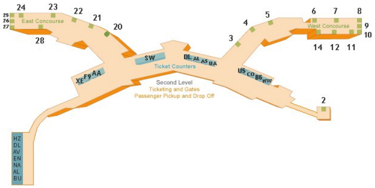
- Optimal solution = 258
- Optimal permutation=[2 4 5 3 1]

$$A = \begin{bmatrix} 0 & 3 & 6 & 4 & 2 \\ 3 & 0 & 2 & 3 & 3 \\ 6 & 2 & 0 & 3 & 4 \\ 4 & 3 & 3 & 0 & 1 \\ 2 & 3 & 4 & 1 & 0 \end{bmatrix} \qquad B_{24531} = \begin{bmatrix} 0 & 6 & 0 & 5 & 10 \\ 6 & 0 & 5 & 4 & 0 \\ 0 & 5 & 0 & 2 & 7 \\ 5 & 4 & 2 & 0 & 15 \\ 10 & 0 & 7 & 15 & 0 \end{bmatrix}$$

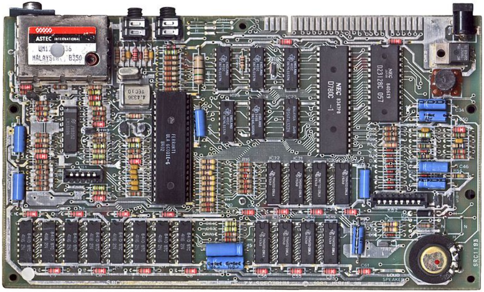
- Hospital Layout - German university hospital, Klinikum Regensburg, built 1972
- Optimality proven in the year 2000
- [Krarup and Pruzan(1978)]



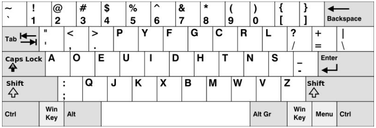
- Airport gate assignment
- Minimize total passenger movement
- Minimize total baggage movement
- [Haghani and Chen(1998)]



- Steinberg wiring problem
- [Steinberg(1961)]
- component placing on circuit boards
- [Rabak and Sichman(2003)]
- Minimizing the number of transistors needed on integrated circuits
- Burkard et al.(1993)



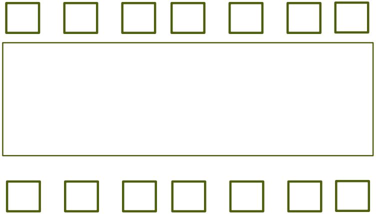
- Optimal placing of letters on keyboards
- Language specific
- Burkard and Offermann(1977)



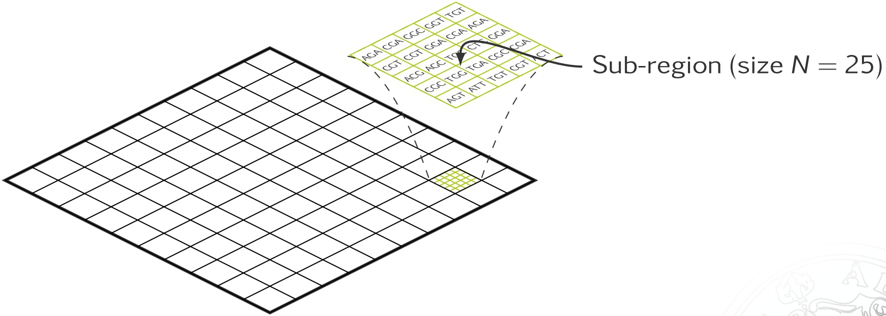
- Optimal placing of letters on touchscreen devices
- Only one or two fingers used
- Dell’Amico et al.(2009)

- Turbine runner in electricity generation
- The weight of the blades can differ up to ±5%
- Objective: To balance the turbine runner
- [Laporte and Mercure(1988)]

- Seating order at tonight's dinner



- Microarrays can have up to 1.3 million probes
- Small subregions can be solved as QAPs
- Objective: To reduce the risk of unintended illumination of probes



- Bandwith minimization of a graph
- Image processing
- Economics
- Molecular conformations in chemistry
- Scheduling
- Supply Chains
- Manufacturing lines

In addition to all the above, many well known problems in combinatorial optimization can be written as QAPs, *e.g.*

- Traveling salesman problem
- Maximum cut problem

Three Objective Functions

- Koopmanns-Beckmann

$$\min A \cdot XBX^T \tag{1}$$

- SDP

$$\min \operatorname{tr}(AXBX^T) \tag{2}$$

- DLR

$$\min XA \cdot BX \tag{3}$$

Koopmans Beckmann form

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} a_{ij} b_{kl} \cdot x_{ik} x_{jl}$$

$$\sum_{i=1}^{N} x_{ij} = 1, \qquad j = 1,...,N;$$

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$$x_{ij} \in \{0,1\}, \qquad i,j = 1,...,N;$$

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- This formulation has $N^2(N-1)^2$ bilinear terms.

min $XA \cdot BX$

$$\min \sum_{i=1}^{N} \sum_{j=1}^{N} a'_{ij} b'_{ij}$$

$$a'_{ij} = \sum_{k=1}^{n} a_{kj} x_{ik} \qquad \forall i,j$$

$$b'_{ij} = \sum_{k=1}^{n} b_{ik} x_{kj} \qquad \forall i,j$$

$$\mathsf{A} = \begin{bmatrix} 0 & 3 & \mathbf{5} & 9 & 6 \\ 3 & 0 & \mathbf{2} & 6 & 9 \\ 5 & 2 & \mathbf{0} & 8 & 10 \\ 9 & 6 & \mathbf{8} & 0 & 2 \\ 6 & 9 & \mathbf{10} & 2 & 0 \end{bmatrix} \qquad \mathsf{B} = \begin{bmatrix} 0 & 4 & 3 & 7 & 7 \\ \mathbf{4} & \mathbf{0} & \mathbf{4} & \mathbf{10} & \mathbf{4} \\ 3 & 4 & 0 & 2 & 3 \\ 7 & 10 & 2 & 0 & 4 \\ 7 & 4 & 3 & 4 & 0 \end{bmatrix}$$

$\min XA \cdot BX$

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$$a'_{23} = 5x_{21} + 2x_{22} + 0x_{23} + 8x_{24} + 10x_{25}$$

$$b'_{23} = 4x_{13} + 0x_{23} + 4x_{33} + 10x_{43} + 4x_{53}$$

Discrete Linear Reformulation (DLR)

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{m=1}^{M_i} B_i^m z_{ij}^m$$

$$\left.\begin{aligned} & z_{ij}^m \leq \overline{A}_j \sum_{k \in \mathbf{K}_i^m} x_{kj} \qquad m = 1, \ldots, M_i \\ & \sum_{m=1}^{M_i} z_{ij}^m = a'_{ij} \end{aligned}\right\} \forall i, j$$

Example for one bilinear term $a'_{23} b'_{23}$

$$a'_{23} = 5x_{21} + 2x_{22} + 0x_{23} + 8x_{24} + 10x_{25}$$

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$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 1$$

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$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 1$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 1$$

$$4z_{23}^1 + 10z_{23}^2$$

$$z_{23}^1 \leq 10(x_{13} + x_{33} + x_{53})$$

$$z_{23}^1 + z_{23}^2 = a'_{23}$$

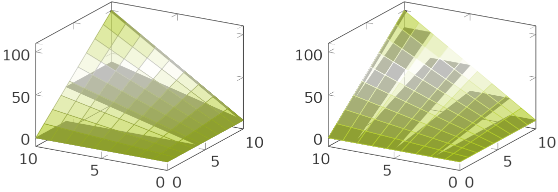

Figure 1: Bilinear term $a'_{23}b'_{23}$ discretized in b'_{23} (to the left) and in a'_{23} (to the right)

- The size of the MILP problem is dependent on the number of unique elements per row.
- Tightness of the MILP problem is dependent on the differences between the elements in each row.

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- A can be modified to any matrix $\tilde{A}$, where $\tilde{a}_{ij} + \tilde{a}_{ji} = a_{ij} + a_{ji}$.
- By solving an LP a priori, we can decrease the model size, tighten the formulation and improve the lower bound.

- The size of the model is dependent on the number of unique elements per row.
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- By solving an LP a priori, we can decrease the model size, tighten the formulation and improve the lower bound.

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 4 & 4 & 5 \\ 1 & 0 & 1 & 1 & 2 & 3 & 3 & 4 \\ 2 & 1 & 0 & 2 & 1 & 2 & 2 & 3 \\ 2 & 1 & 2 & 0 & 1 & 2 & 2 & 3 \\ 3 & 2 & 1 & 1 & 0 & 1 & 1 & 2 \\ 4 & 3 & 2 & 2 & 1 & 0 & 2 & 3 \\ 4 & 3 & 2 & 2 & 1 & 2 & 0 & 1 \\ 5 & 4 & 3 & 3 & 2 & 3 & 1 & 0 \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} 0 & 2 & 2 & 2 & 6 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 \\ 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 \\ 4 & 4 & 4 & 4 & 4 & 4 & 0 & 0 \end{bmatrix}$$

- Does not break the symmetries in the problem.

Instance	Size	opt	old LB	DLR	Time(minutes)
esc32a	32	130	103	130	1964
esc32b	32	168	132	168	3500
esc32c	32	642	616	642	254
esc32d	32	200	191	200	10
esc64a	64	116	98	116	48

Table 1: Solution times for the instances esc32a, esc32b, esc32c, esc32d and esc64a from the QAPLIB to optimality using Gurobi 4.1 with default settings.

- Instances presented in 1990 and solved 2010 with our models

A few references

Ali Haghani and Min-Ching Chen.
Optimizing gate assignments at airport terminals.
Transportation Research Part A: Policy and Practice, 32(6):437 – 454, 1998.

Jakob Krarup and Peter Mark Pruzan.
Computer-aided layout design.
In M.L. Balinski and C. Lemarechal, editors, *Mathematical Programming in Use*, volume 9 of *Mathematical Programming Studies*, pages 75–94. Springer Berlin Heidelberg, 1978.

Gilbert Laporte and Helene Mercure.
Balancing hydraulic turbine runners: A quadratic assignment problem.
European Journal of Operational Research, 35(3):378 – 381, 1988.

Axel Nyberg and Tapio Westerlund.
A new exact discrete linear reformulation of the quadratic assignment problem.
European Journal of Operational Research, 220(2):314 – 319, 2012.

C.S Rabak and J.S Sichman.
Using a-teams to optimize automatic insertion of electronic components.
Advanced Engineering Informatics, 17(2):95 – 106, 2003.

Leon Steinberg.
The backboard wiring problem: A placement algorithm.
SIAM Review, 3(1):37–50, January 1961.

Thank you for listening!

Questions?